the solution of which is

$$c(t) = \frac{K_1 e^{-K_1 \omega_r t}}{K_1 + K_2 - K_2 e^{-K_1 \omega_r t}} \tag{16}$$

The average concentration is

$$\bar{c} = \int_0^\infty c(t) e^{-\omega_r t} d\omega_r t \tag{17}$$

The substitution of $u = e^{-\omega_r t}$ transforms Equation (17) into a convenient form for numerical integration

$$c = \int_0^1 \frac{K_1 \, u^{K_1} \, du}{K_1 + K_2 - K_2 \, u^{K_1}} \tag{18}$$

Figure 2 presents calculated results for the selectivity of the first-order reaction path based on Equations (13) and (18). Equation (18) was numerically integrated by using Simpson's rule with equal step sizes of 0.01 for Δu. For values of the second-order rate modulus K_2 of the same order of magnitude or greater than K_1, there is a significant difference in the selectivity of the first-order reaction path for zero and for infinite drop interaction rates. The effect indicated is an increase in selectivity for a high rate of drop interaction. This is the trend which would be expected, based on the solutions for a single reaction path, since the first-order reaction is unaffected by the drop interaction, while the second-order reaction is suppressed by it. The effects become more pronounced at large values of both K_1 and K_2.

A convenient summary of analytic results for zero-order–first-order, zero-order–second-order, and the above case of first-order–second-order parallel competing reaction paths is presented in Table 1 for the limiting cases of zero and infinite drop interaction rates. The computer program for dispersed phase mixing is easily modified to include the general case of competing parallel reaction with an arbitrary drop interaction rate. Selected results for $K_2 = 10$, $I = 10$, and $I = 100$ appear in Figure 2. It is evident from the figure that the influence of the coalescence and redispersion is slight for this example unless both reactions have sizable values for reaction rate moduli.

While the most general problem of reaction of several reactant species has not been solved, the solutions presented here in conjunction with those of references *1* and *4* should provide guidance for design in situations involving competitive reactions.

NOTATION

A_n = apparent decay constant
c = concentration variable
f = conversion
I = drop interaction modulus
k = reaction rate constant
K = reaction rate modulus
n = discrete concentration variable
N = number of concentration intervals
$p(c)$ = density function for drops
s = reaction order
S = selectivity
t = time, hr.
ω_i = drop interaction frequency, hr.$^{-1}$
ω_r = reciprocal residence time, hr.$^{-1}$

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Drag Augmentation by Polymer Addition

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A considerable body of recent literature (*1* to *7*) has demonstrated that the addition of small quantities of very high molecular weight polymers can lead to a drastic reduction in the skin friction associated with a given turbulent flow. An obvious application of this phenomenon is in the field of torpedo technology, and the experiments reported here were so motivated.

The experimental equipment was exceedingly simple and consisted of an open column, 17 ft. high and 8.5 in. in diameter, which could be filled with a suitable solution. Test bodies were placed on the bottom of this tank and then accelerated to terminal velocity by a system of counterweights. An arrangement employing photocells was used to determine the terminal velocity thus attained. The polymer chosen for addition was one well known for its drag-reducing capability, polyethylene oxide, [Union Carbide Polyox (code WSR 301)] with a molecular weight of about 4.5×10^{-6}. The concentrations investigated were 0, 20, 50, and 100 p.p.m. The solutions were essentially Newtonian with an intrinsic viscosity of 16.5 dl./g. Even for the most concentrated solution, the relative viscosity was less than 1.2. Two test bodies actually employed were a flat plate and a scale model of a torpedo. In both cases, calculations indicated that for pure solvent, the boundary layer on the bodies would be turbulent, but that approximately half of the drag experienced by the torpedo would be due to form drag.

As expected, the polymer addition led to a substantial increase in the terminal velocity achieved by the flat plate, that is, the drag on the plate was significantly reduced. At 20 p.p.m. the terminal velocity of the torpedo was also increased, but at higher concentrations, it had

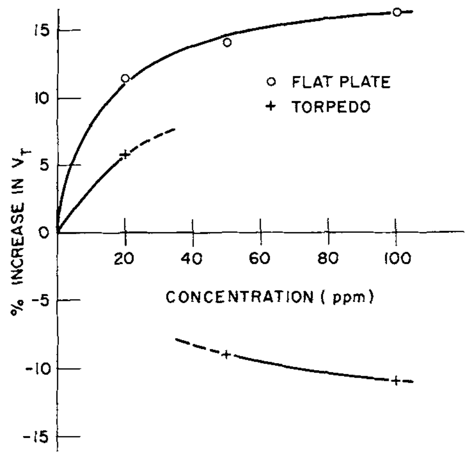

Fig. 1. Effect of additive concentration upon terminal velocity.

a lower terminal velocity! That is, the torpedo suffered a drag augmentation due to polymer addition. These data are presented in Figure 1.

Presumably the resolution of this disparity lies in the role of form drag. It seems likely that the addition of polymer so laminarized the boundary layer as to cause early separation and thus an increased form drag. In short, these experiments provide, in a modern context, a classic illustration of Eiffel's paradox (*8*).

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Remarks on the Withdrawal Problem for Ellis Fluids

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In a recent paper, Matsuhisa and Bird (*8*) have presented several solutions to non-Newtonian flow problems by using the Ellis model in the following half tau form:

$$\frac{du}{dy} = -\frac{\tau_{yx}}{\eta_o}\left[1 + \left|\frac{\tau_{yx}}{\tau_H}\right|^{\alpha-1}\right] \tag{1}$$

They discussed the withdrawal problem (*5*) in which a solid wall immersed in a liquid moves upward with constant velocity. Predictions of film thickness were developed as a function of rheological constants, wall velocity (u_w), and distance (z) from the free surface of the liquid. Their prediction for the unsteady state case is given by their Equation (29); in the notation of this communication, it is

$$\frac{z}{u_w t} = 1 - \frac{\rho g h^2}{\eta_o u_w}\left[1 + \left(\frac{\rho g h}{\tau_H}\right)^{\alpha-1}\right] \tag{2}$$

For the steady state case, their solution is

$$\frac{\rho g h_o^2}{\eta_o u_w}\left[1 + \left(\frac{\rho g h_o}{\tau_H}\right)^{\alpha-1}\right] = 1 \tag{3}$$

The main purpose of this communication is to note that, according to available evidence, these solutions do not properly predict film thicknesses that occur in *withdrawal*; they are, however, applicable for the less complex phenomena of *drainage*. In other words, Equations (2) and (3) are better described as solutions of the drainage problem.

A secondary purpose of this note is to compare these equations with Ellis fluid drainage equations derived elsewhere (*5*). A third purpose, related to the first two, is to present some minor disadvantages of the half tau form of the Ellis model.

The solutions given by Equations (2) and (3) consider only gravitational and viscous forces. However, surface tension is also important in withdrawal, as shown theoretically from 1942 on by Russian workers (*6*, *3*, *7*) and experimentally from 1946 on by workers in The Netherlands (*2*, *9*).

The inapplicability of Equations (2) and (3) for withdrawal can be shown most easily by comparing steady state solutions for the limiting case of a Newtonian fluid. As shown in Appendix A, Equation (3) reduces to

$$h_o\left(\frac{\rho g}{\mu u_w}\right)^{1/2} = 1 \tag{4}$$

On the other hand, the best prediction of film thickness to date has been found, upon consideration of surface tension, to be

$$h_o\left(\frac{\rho g}{\mu u_w}\right)^{1/2} = 0.944\left(\frac{\mu u_w}{\sigma}\right)^{1/6}\left(1 - \frac{h_o^2 \rho g}{\mu u_w}\right)^{2/3} \tag{5}$$